A targeted implicit particle filter

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Bayes' theorem

$\mathbf{x} \in \mathcal{R}^{N_x}$ Model variables

$\mathbf{y} \in \mathcal{R}^{N_y}$ Observations

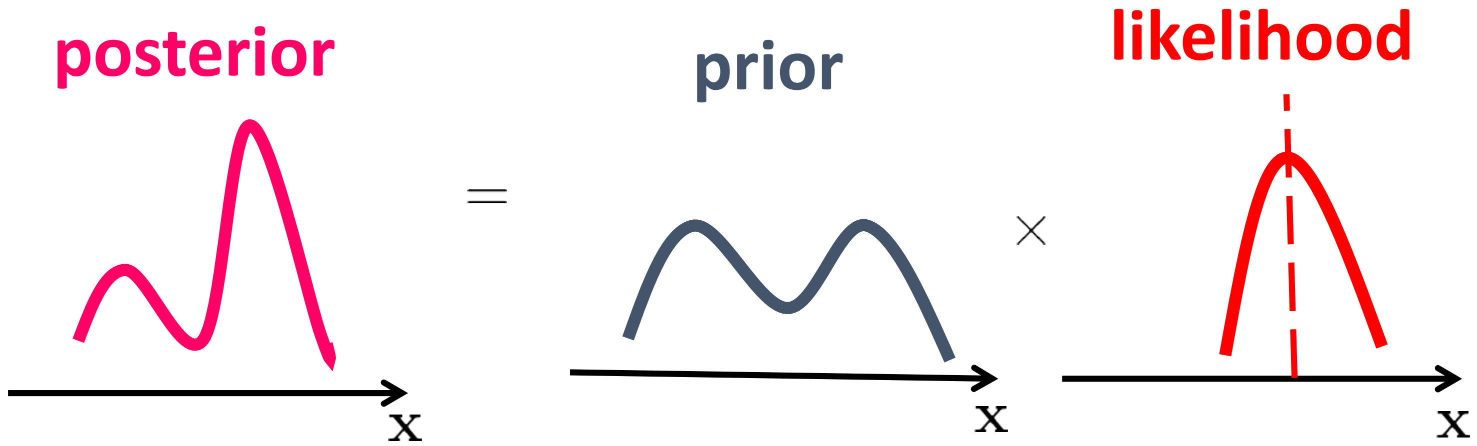
$\mathbf{x}^0 \quad r.v.$

$\mathbf{x}^t = f(\mathbf{x}^{t-1}) + \boldsymbol{\beta}^t$

$\mathbf{y}^t = h(\mathbf{x}^t) + \boldsymbol{\eta}^t$

To obtain the posterior:

$$p(\mathbf{x}^t|\mathbf{y}^t) = \frac{1}{C} p(\mathbf{x}^t) p(\mathbf{y}^t|\mathbf{x}^t)$$

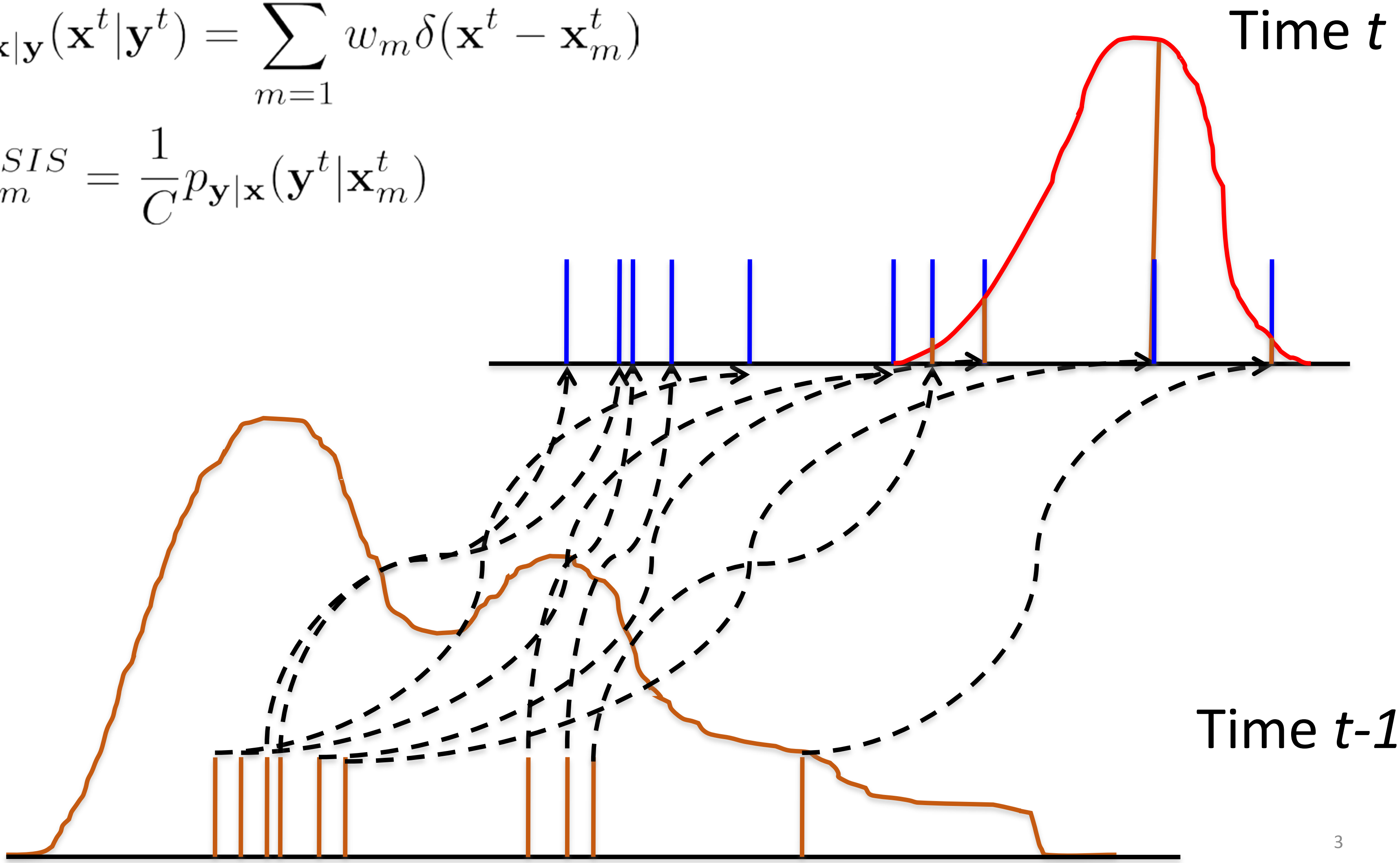


Simple importance sampling

$$p(\mathbf{x}^t|\mathbf{y}^t) = \frac{1}{C}p(\mathbf{y}^t|\mathbf{x}^t)p(\mathbf{x}^t)$$

$$p_{\mathbf{x}|\mathbf{y}}(\mathbf{x}^t|\mathbf{y}^t) = \sum_{m=1}^{M} w_m \delta(\mathbf{x}^t - \mathbf{x}_m^t)$$

$$w_m^{SIS} = \frac{1}{C} p_{\mathbf{y}|\mathbf{x}}(\mathbf{y}^t|\mathbf{x}_m^t)$$

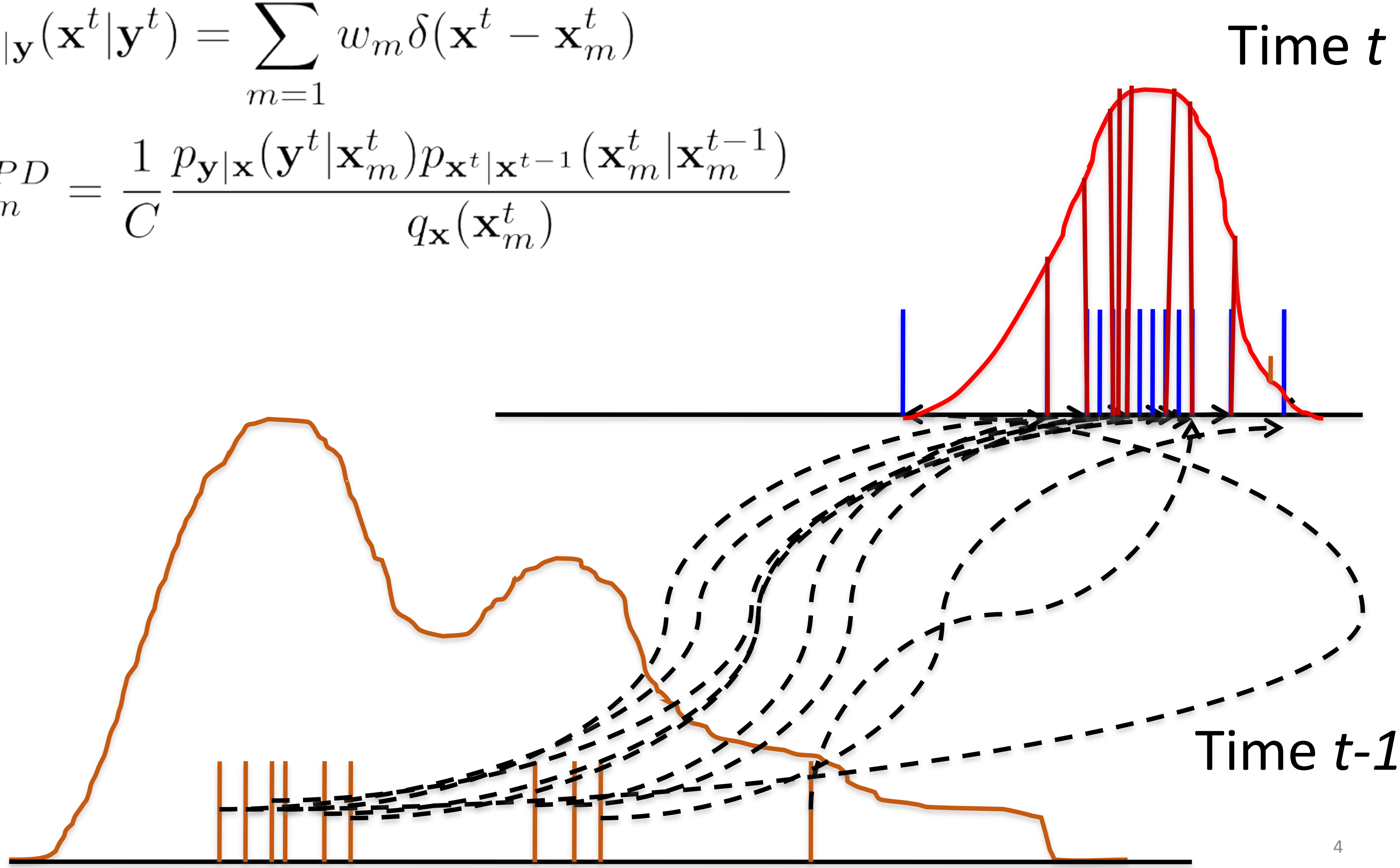


Proposal densities

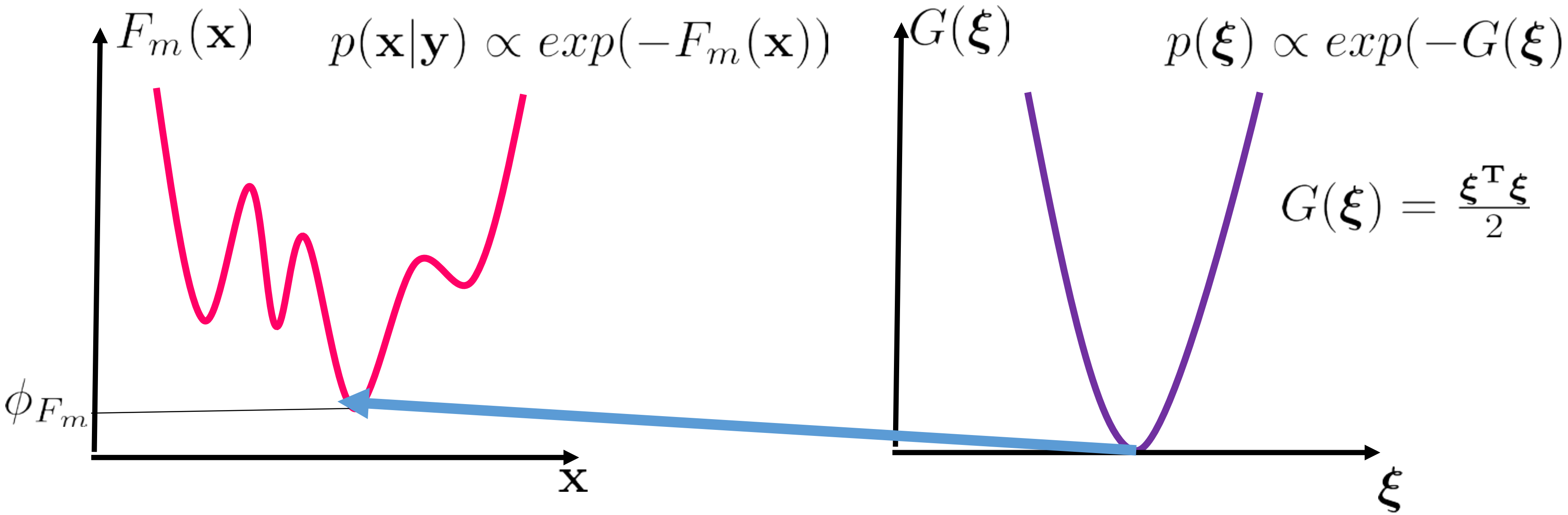
$$p(\mathbf{x}^t|\mathbf{y}^t) = \frac{1}{C} p(\mathbf{y}^t|\mathbf{x}^t) \frac{p(\mathbf{x}^t)}{q(\mathbf{x}^t)} q(\mathbf{x}^t)$$

$$p_{\mathbf{x}|\mathbf{y}}(\mathbf{x}^t|\mathbf{y}^t) = \sum_{m=1}^{M} w_m \delta(\mathbf{x}^t - \mathbf{x}^t_m)$$

$$w_m^{PD} = \frac{1}{C} \frac{p_{\mathbf{y}|\mathbf{x}}(\mathbf{y}^t|\mathbf{x}^t_m) p_{\mathbf{x}^t|\mathbf{x}^{t-1}}(\mathbf{x}^t_m|\mathbf{x}^{t-1}_m)}{q_{\mathbf{x}}(\mathbf{x}^t_m)}$$



The implicit particle filter (Chorin et al, 2010…)



$$F = \cancel{J_{back}} + J_{model} + J_{obs}$$

$$\cancel{||\mathbf{x}^0 - \mathbf{x}^{0,b}||^2_{\mathbf{B}^{-\frac{1}{2}}}} + ||\mathbf{x}^t - f(\mathbf{x}^{t-1})||^2_{\mathbf{Q}^{-\frac{1}{2}}} + ||\mathbf{y}^t - h(\mathbf{x}^t)||^2_{\mathbf{R}^{-\frac{1}{2}}}$$

$$F_m = J_{model,m} + J_{obs,m}$$

Modified weak-constraint 4DVar, as suggested by van Leeuwen 2009

The implicit particle filter (Chorin et al, 2010…)

Posterior representation

$$p(\mathbf{x}|\mathbf{y}) = \sum_{m=1}^{M} w_m \delta(\mathbf{x} - g(\boldsymbol{\xi}_m))$$

$$g: \quad \boldsymbol{\xi} \rightarrow \mathbf{x}$$

$$J_m = \left. \frac{\partial g(\boldsymbol{\xi}^t)}{\partial \boldsymbol{\xi}^t} \right|_{\boldsymbol{\xi}^t_m} \in \mathcal{R}^{N_x \times N_x}$$

Weights

Implicit equation

$$\underbrace{w_m^{IPF} \propto \frac{exp[-F_m(\mathbf{x}^t_m)]}{exp[-G(\boldsymbol{\xi}^t_m)]} |det(J_m)| \qquad F_m(\mathbf{x}_m) - \phi_{F_m} = G(\boldsymbol{\xi}_m) - \phi_{G_m}}$$

$$w_m^{IPF} \propto exp[-\phi_{F_m}] \, |det(J_m)|$$

The targeted implicit particle filter

Posterior representation $$p(\mathbf{x}|\mathbf{y}) = \sum_{m=1}^{M} w_m \delta(\mathbf{x} - g(\boldsymbol{\xi}_m))$$

$$w_m^{TIPF} \propto exp[-F_m(\mathbf{x}_m^t)]\,|det(J_m)| = constant \ \forall m$$

Map: $$\mathbf{x}_m = \boldsymbol{\mu}_m + \lambda_m \boldsymbol{\Gamma}_m^{\mathbf{T}} \boldsymbol{\xi}_m$$

(similar to van Leeuwen, Chorin)

Jacobian: $$J_m = \lambda_m \boldsymbol{\Gamma}_m^{\mathbf{T}} \left(\mathbf{I} + \frac{\boldsymbol{\xi}_m}{\lambda_m} \nabla_{\boldsymbol{\xi}} \lambda_m \right)$$

$$det(J_m) = \lambda_m^{N_x} det\left(\boldsymbol{\Gamma}_m^{\mathbf{T}}\right) \left(1 + \nabla_{\boldsymbol{\xi}} \lambda_m \frac{\boldsymbol{\xi}_m}{\lambda_m}\right)$$

The targeted implicit particle filter

Solution to make $w_m \propto exp[-constant]$

$$\boxed{constant = F_m(\mathbf{x}) - G(\boldsymbol{\xi}) - N_x\lambda_m - ln\left(1 + \nabla_{\boldsymbol{\xi}}\lambda_m \frac{\boldsymbol{\xi}}{\lambda_m}\right)}$$

Simple case:

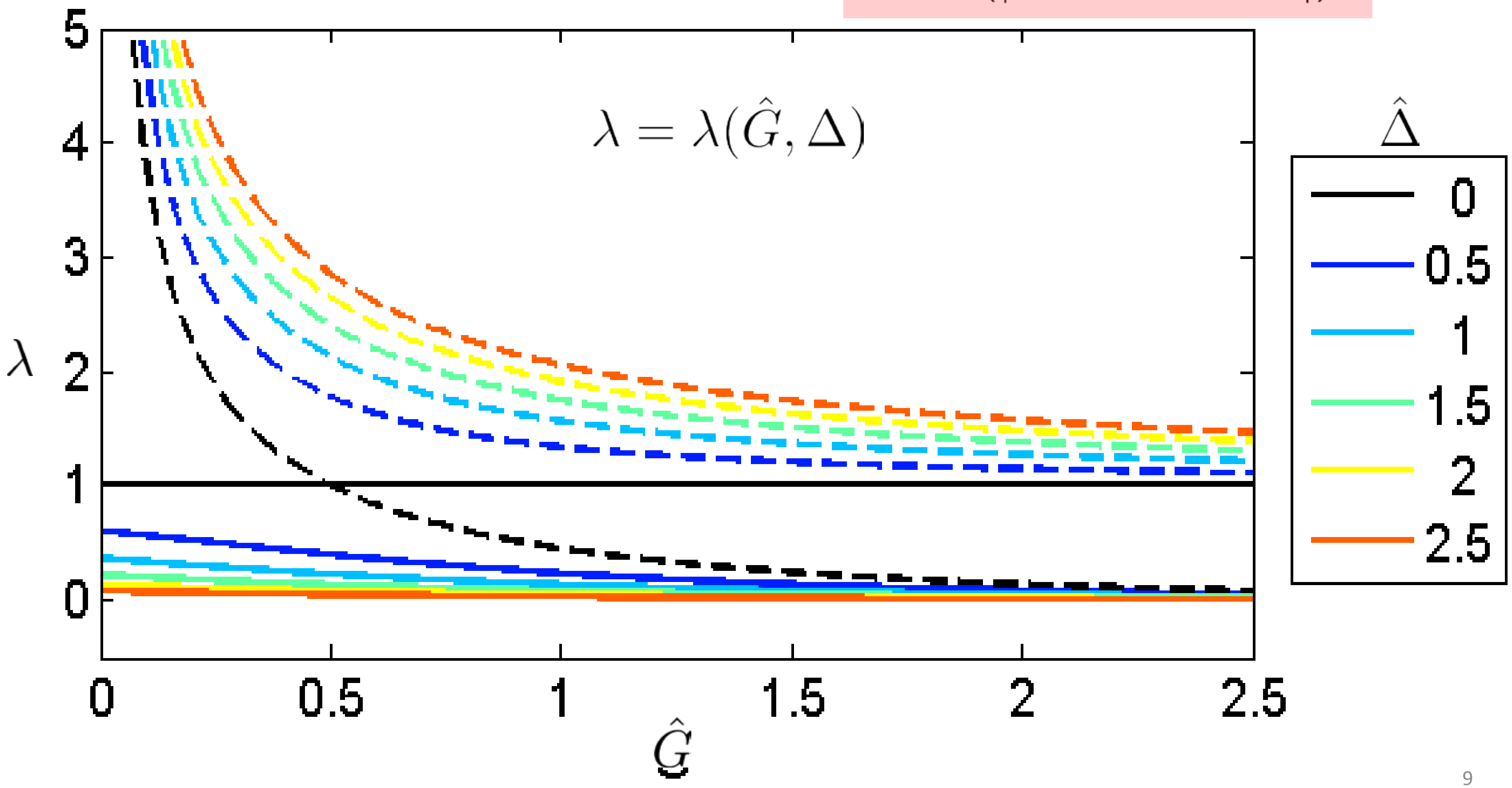
1 step, Gaussian obs and model error. $\boldsymbol{\Gamma}_m = \mathbf{A}^{1/2}$

$$\Delta = C - \phi_{F_m} \Rightarrow \underbrace{\Delta}_{O(N_y)} = \underbrace{(\lambda_m^2 - 1)G(\boldsymbol{\xi})}_{O(N_x)} - \underbrace{N_x Ln[\lambda_m]}_{O(N_x)} + \underbrace{Ln\left(\left|1 + \frac{2G}{\lambda_m}\frac{\partial\lambda_m}{\partial G}\right|\right)}_{< O(Ln(N_x))}$$

$$\boxed{\hat{\Delta} = (\lambda_m^2 - 1)\hat{G}(\boldsymbol{\xi}) - Ln[\lambda_m] - \frac{1}{N_x}Ln\left(\left|1 + \frac{2\hat{G}}{\lambda_m}\frac{\partial\lambda_m}{\partial\hat{G}}\right|\right)}$$

Solutions in the mod/obs Gaussian 1-step case

$$\hat{\Delta} = (\lambda_m^2 - 1)\hat{G}(\boldsymbol{\xi}) - Ln[\lambda_m] - \epsilon Ln\left(\left|1 + \frac{2\hat{G}}{\lambda_m}\frac{\partial \lambda_m}{\partial \hat{G}}\right|\right)$$



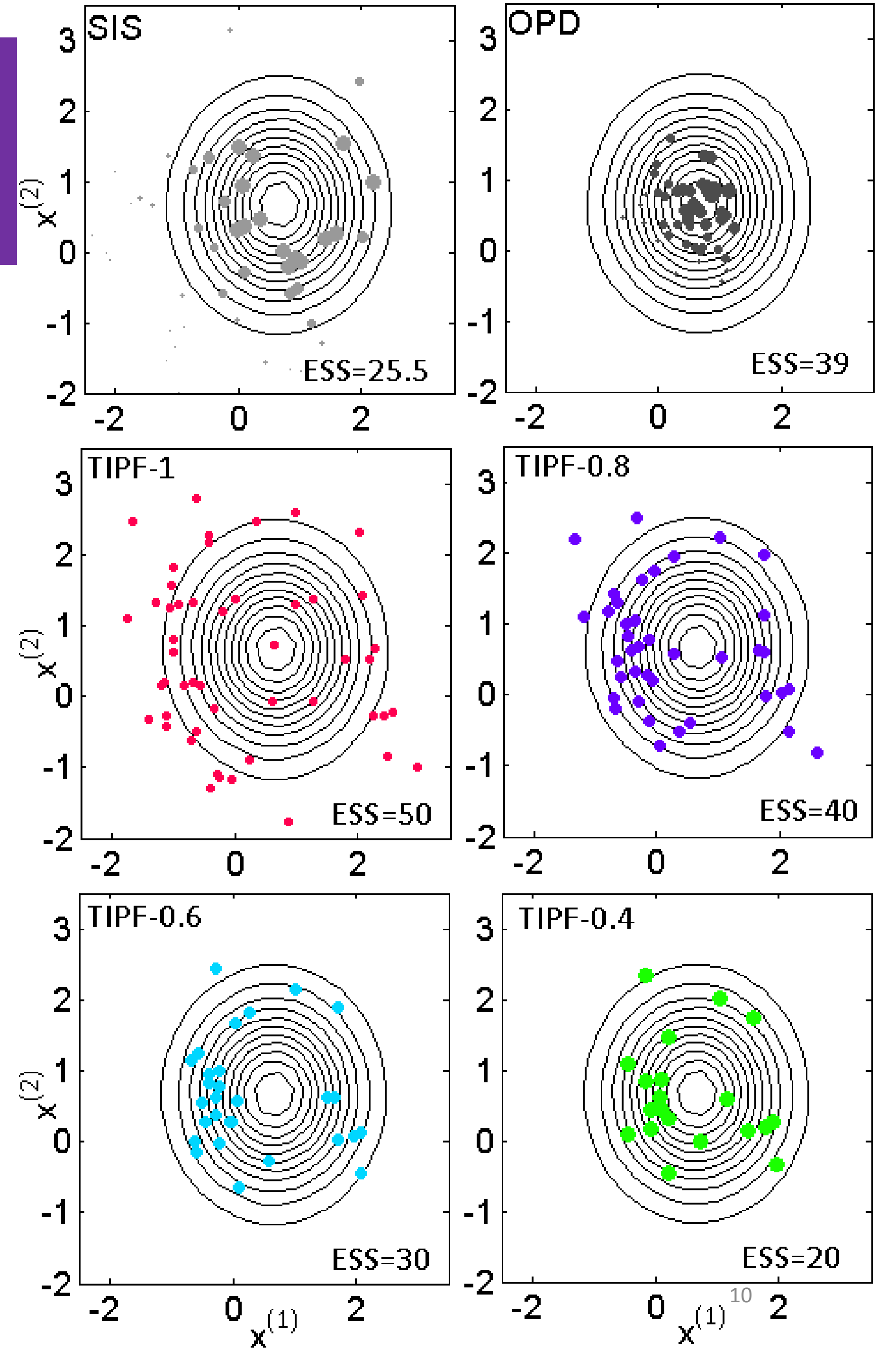
A simple linear system
Synder et al (2010)

Evolution and observation

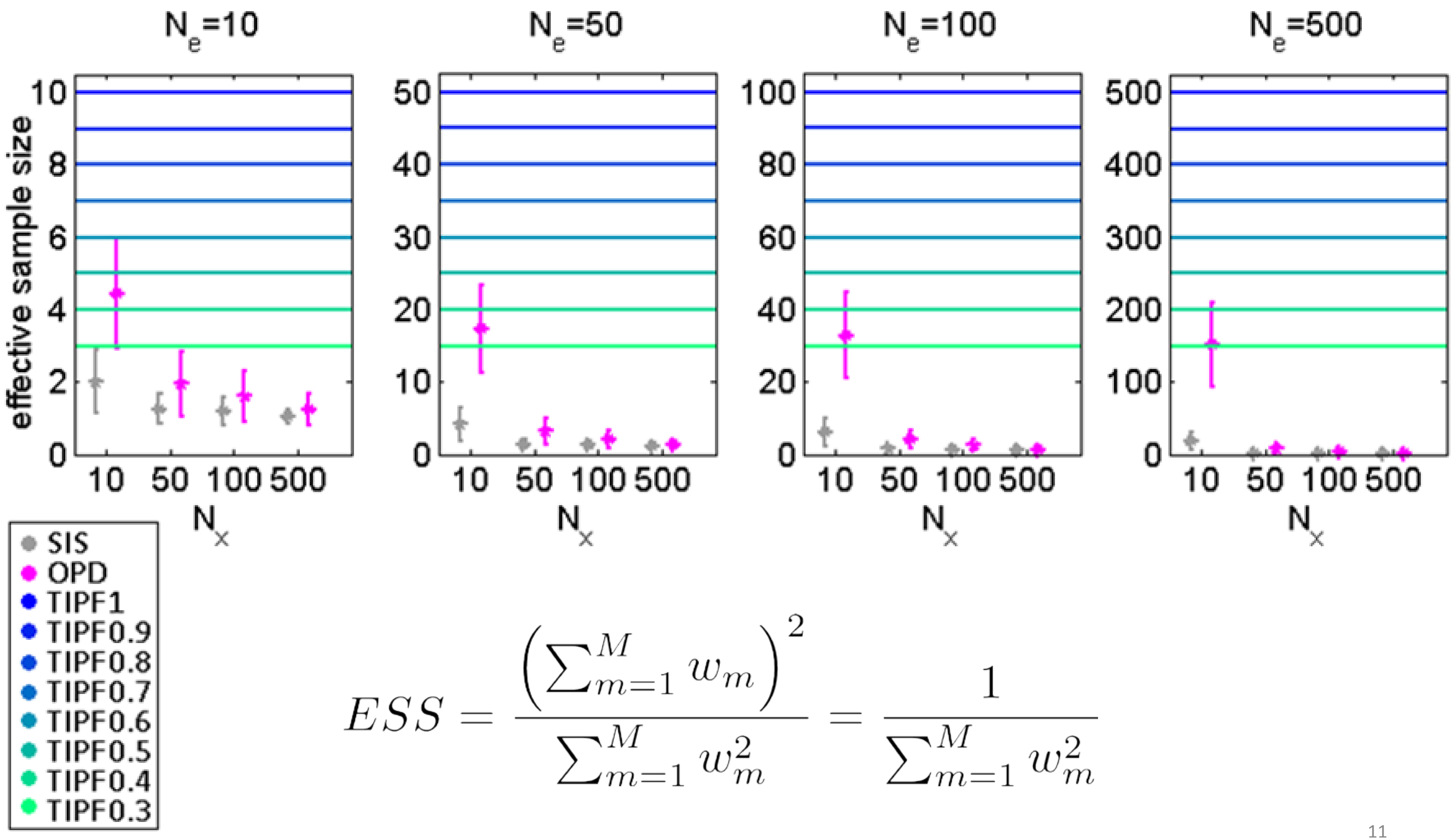
$$\mathbf{x}^1 = \theta \mathbf{x}^0 + \boldsymbol{\beta}^1$$
$$\mathbf{y}^1 = \mathbf{x}^1 + \boldsymbol{\eta}^1$$

System parameters

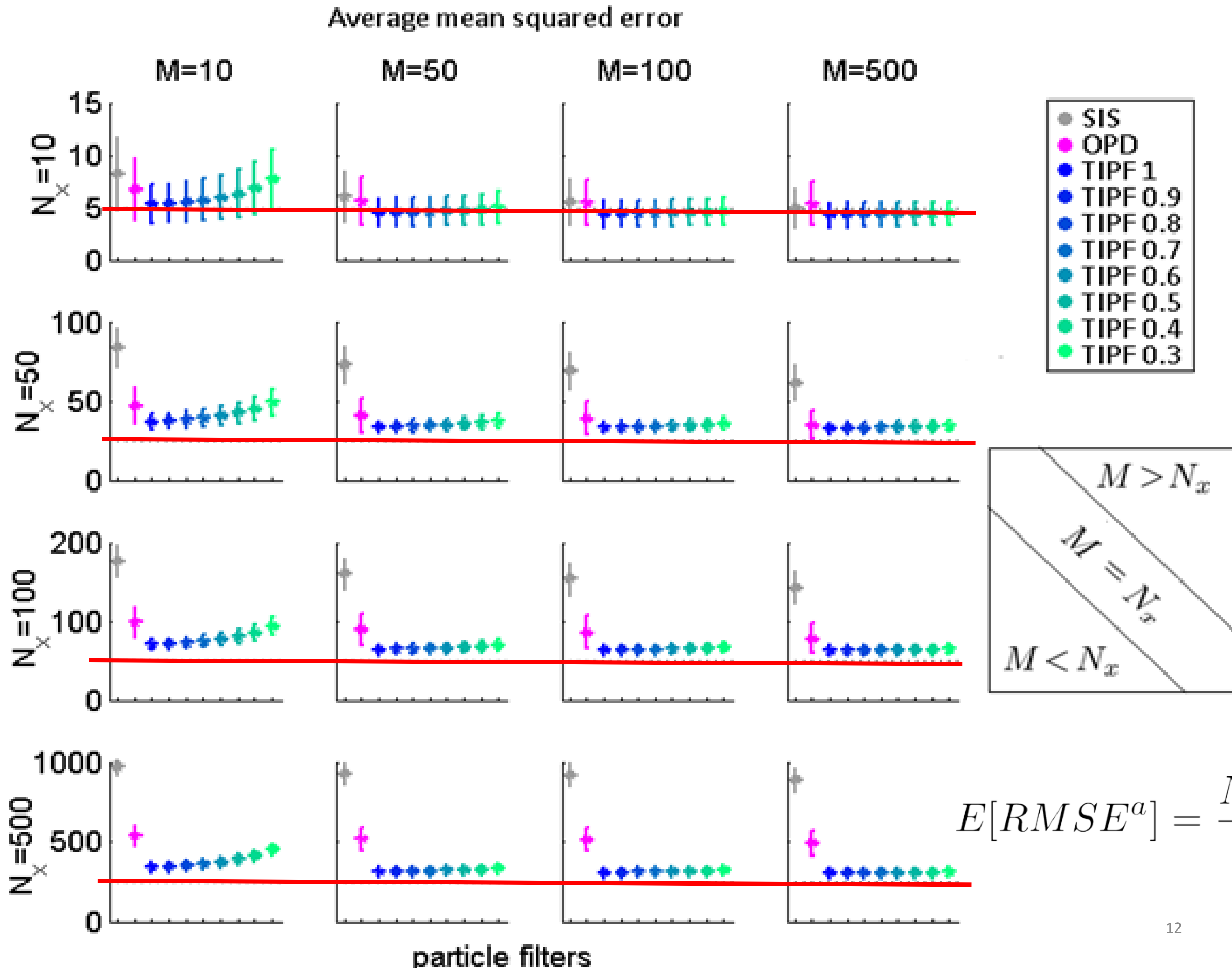
$$\mathbf{x}^0,\ \mathbf{x}^1,\ \mathbf{y}^1,\ \boldsymbol{\beta}^1,\ \boldsymbol{\eta}^1 \ \in \mathcal{R}^{N_x}$$
$$\mathbf{H} = \mathbf{I}, \theta = \frac{1}{\sqrt{2}}$$
$$\mathbf{x}^0 \sim N(\mathbf{0}, \mathbf{B} = \mathbf{I})$$
$$\boldsymbol{\beta}^1 \sim N(\mathbf{0}, \mathbf{Q} = \frac{1}{2}\ \mathbf{I})$$
$$\boldsymbol{\eta}^1 \sim N(\mathbf{0}, \mathbf{Q} = \mathbf{I})$$



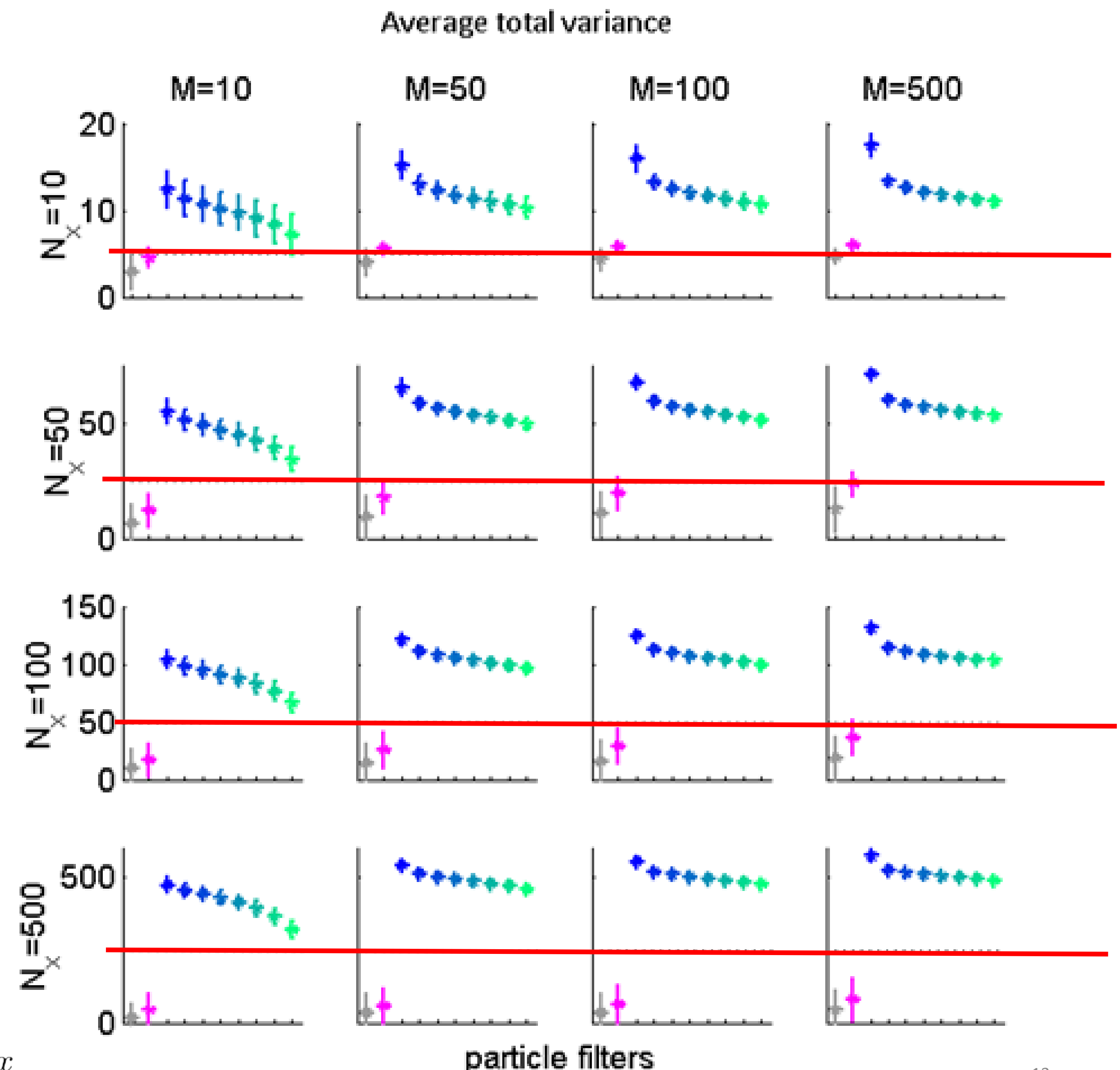
Effective sample sizes

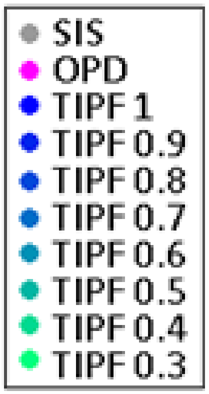


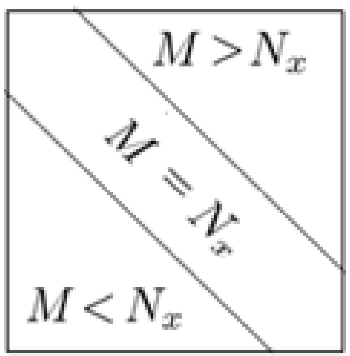
$$ESS = \frac{\left(\sum_{m=1}^{M} w_m\right)^2}{\sum_{m=1}^{M} w_m^2} = \frac{1}{\sum_{m=1}^{M} w_m^2}$$



$$E[RMSE^a] = \frac{N_x}{2}$$

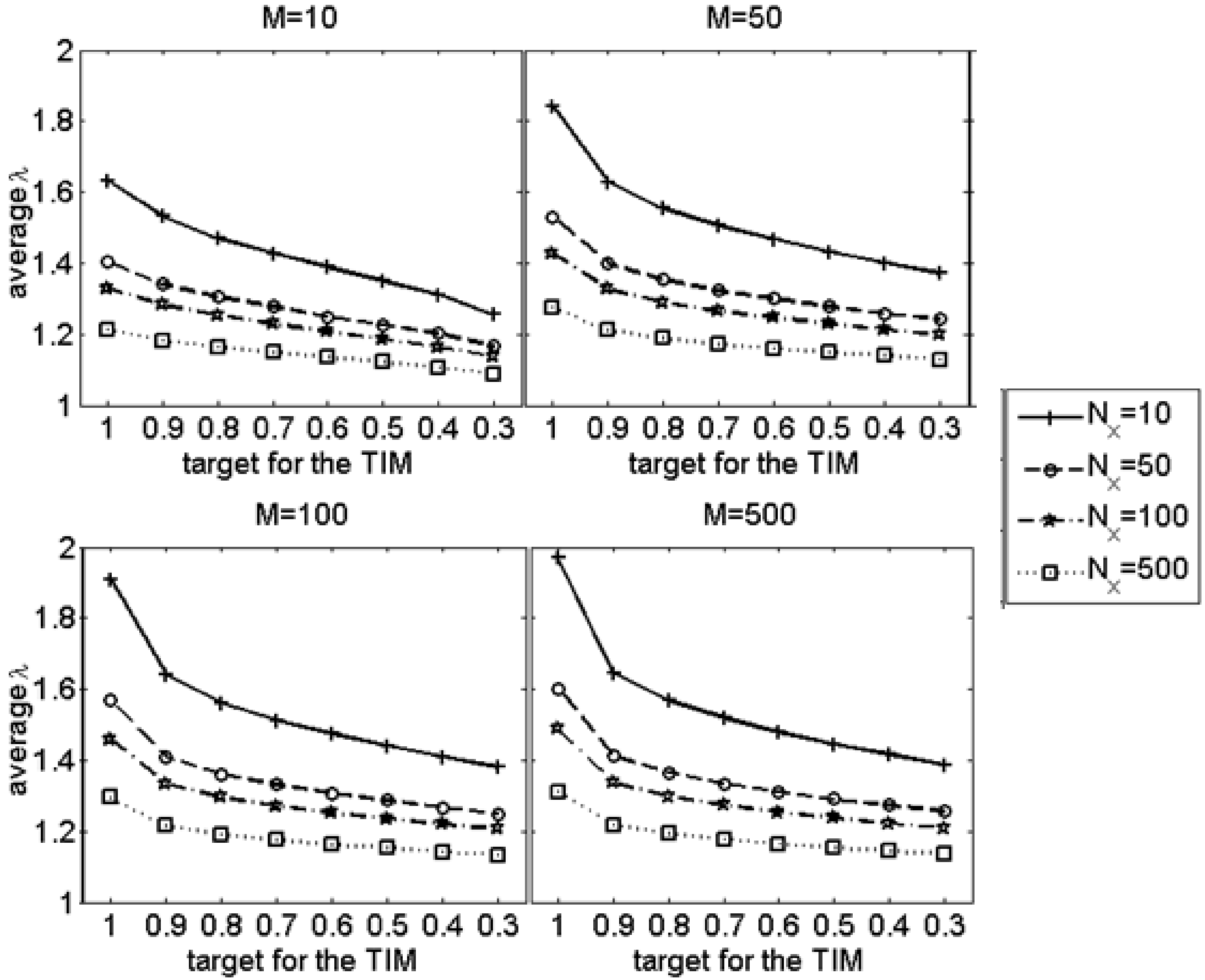






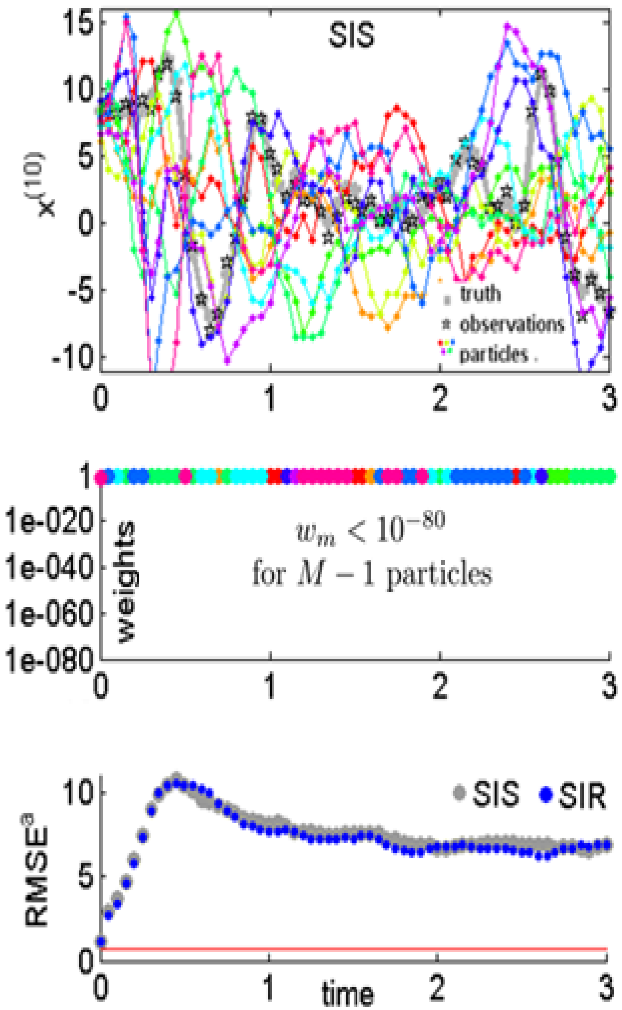
$$E[Tr[P^a]] = \frac{N_x}{2}$$

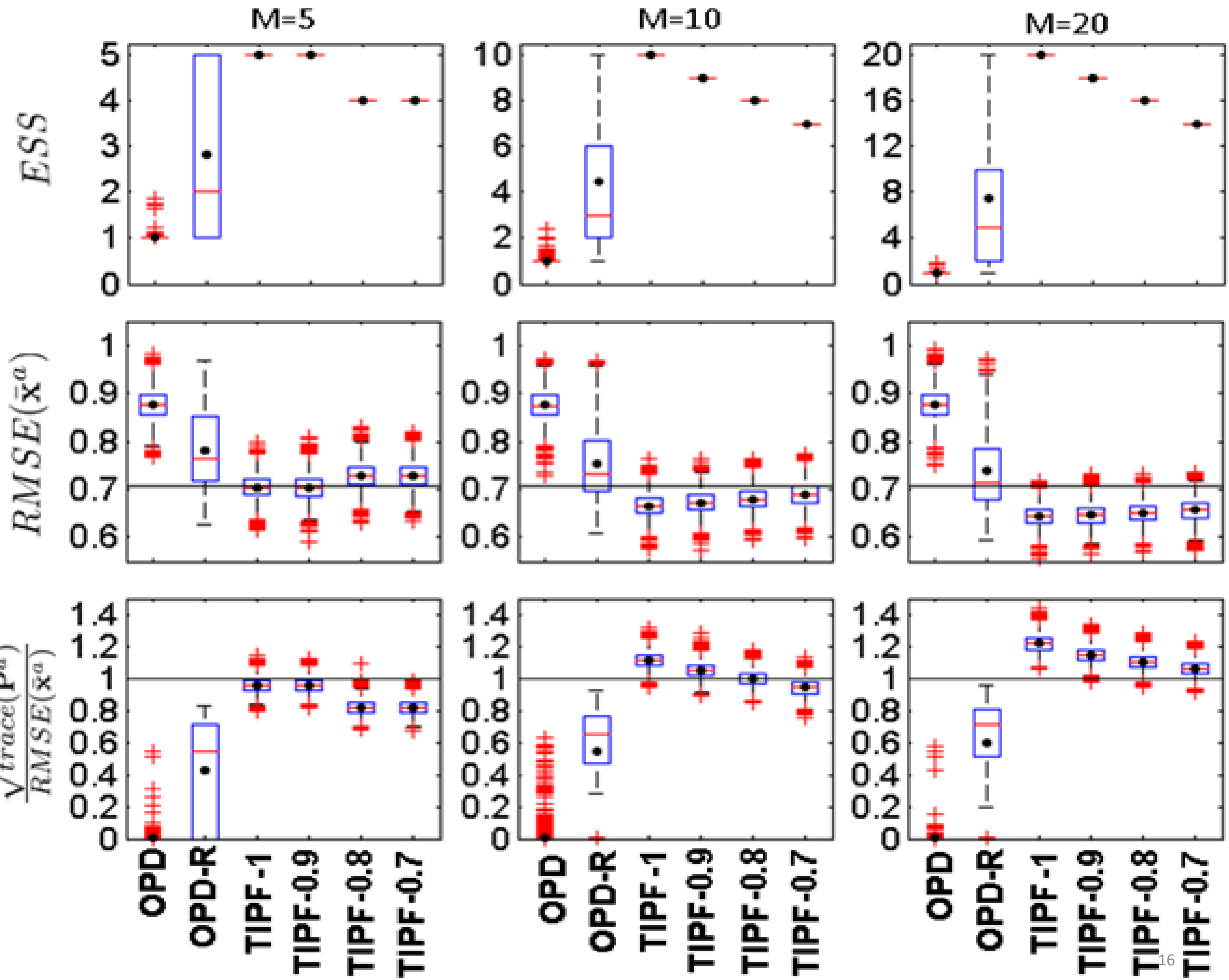
Expansion parameter values



400-variable Lorenz 1996

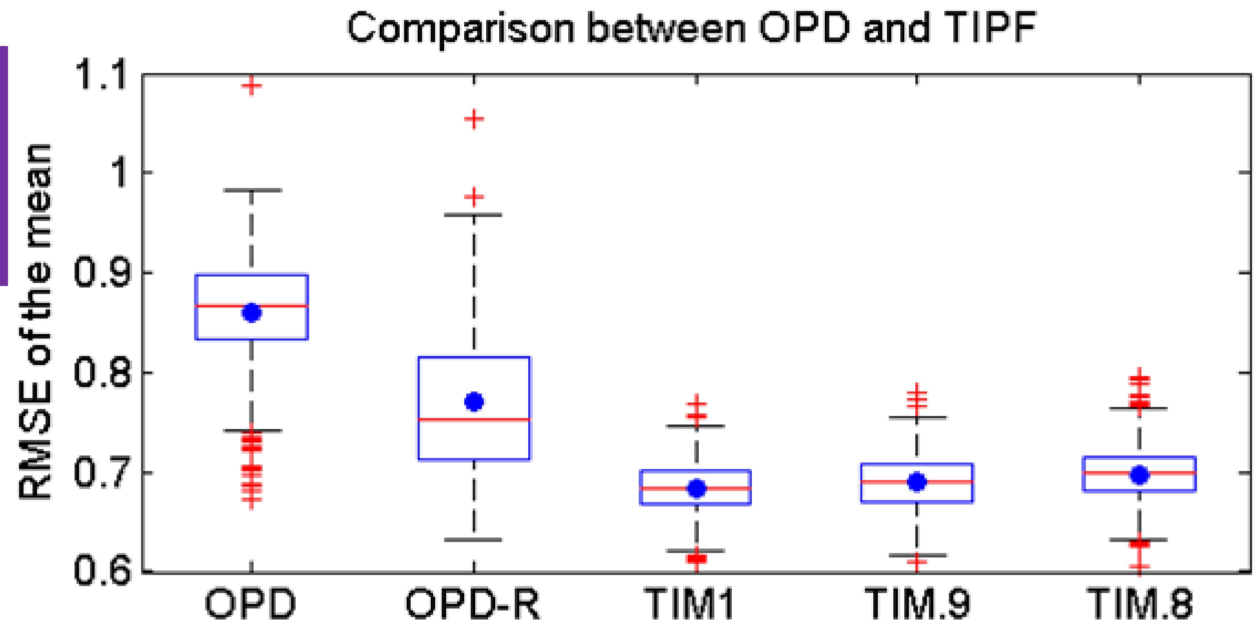
$$\frac{dx_n}{dt} = (x_{n+1} - x_{n-2})x_{n-1} - x_n + \beta_n$$

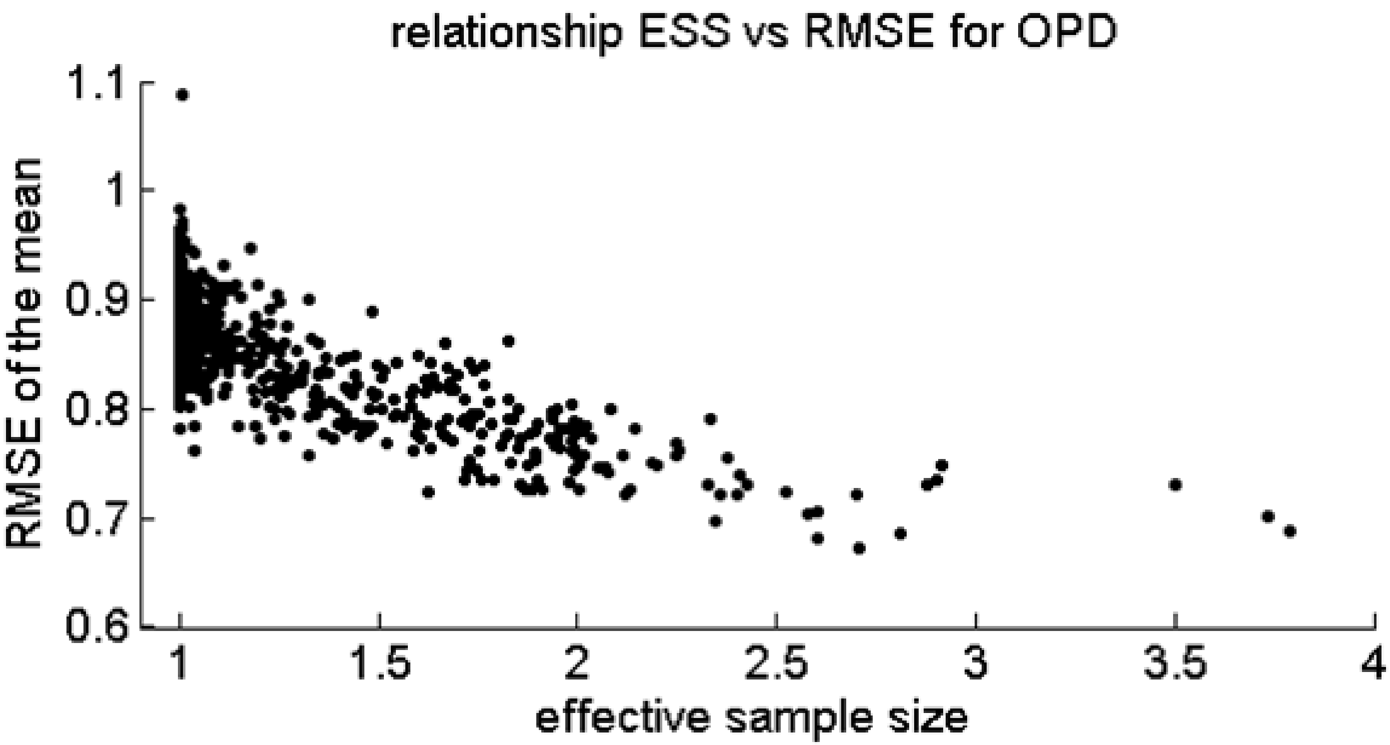




M=5
M=10
M=20
ESS
RMSE($\bar{x}^a$)
$\frac{\sqrt{trace(\mathbf{P}^a)}}{RMSE(\bar{x}^a)}$
OPD
OPD-R
TIPF-1
TIPF-0.9
TIPF-0.8
TIPF-0.7

400-variable
L96 , M=10





Conclusions

Our new particle filter has the following characteristics:
- Explores regions of high posterior probability.
- Has equal weights by construction*.

Next steps:
- Explore the full 4D-Var density (i.e. do not observe and assimilate every step).
- Implement the 2-layer quasi-geostrophic model.