A practical quadratic ensemble filter

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Outline

- Linear filter
- Quadratic filter
- Simplified quadratic filter
- Testing

Linear filter

We only know the distribution of the system state (described by a vector x) and the indirect observations denoted by a vector y. To estimate the true state, we assume a linear relation:

$$\overline{x_a} = \bar{x} + K\left(y - \overline{h[x]}\right)$$

K is a function mapping from the observation space to the system space, usually known as Kalman gain. We find K so that P_a is minimum.

The best estimation of the linear filter

$$\mathrm{K} = P_{xy}\left(P_{yy} + R\right)^{-1}$$

$$P_a = P_{xx} - P_{xy}\left(P_{yy} + R\right)^{-1} P_{xy}^T$$

$$P_{xx} = \overline{(x - \bar{x})(x - \bar{x})^T}$$

$$P_{xy} = \overline{(x - \bar{x})\left(h[x] - \overline{h[x]}\right)^T}$$

$$P_{yy} = \overline{\left(h[x] - \overline{h[x]}\right)\left(h[x] - \overline{h[x]}\right)^T}$$

Role of the system model

Propagate the distribution at the time t=0 to the time t: (x_a, P_a) -> (x_f, P_f)

$$\overline{x_f} = \overline{M[x_a]}$$

$$P_f = \overline{\left(M[x_a] - \overline{M[x_a]}\right)\left(M[x_a] - \overline{M[x_a]}\right)^T} + \mathrm{Q}$$

So the equation for the linear filter should be understood as:

$$\overline{x_a} = \overline{x_f} + K\left(y - \overline{h[x_f]}\right)$$

Varieties of linear filter

- Kalman filter: M is a linear model
- Extended Kalman filter: P_f is approximated using the derivative operator of M (known as tangent linear model)
- Unscented Kalman filter: P_{xx}, P_{xy}, P_{yy} are approximated using a sample of the initial distribution and the exact nonlinear model M.
- Ensemble Kalman filter: similar to UKF in idea but the sample size is small compared to the state dimension (only an error subspace is explored).

Ensemble linear filter

Ensemble in the model space:

$$X = \frac{1}{\sqrt{K-1}}(x - \bar{x})$$

Ensemble in the observation space:

$$Y = \frac{1}{\sqrt{K-1}}(h[x] - \overline{h[x]})$$

Kalman gain:

$$K = XY^T[YY^T + R]^{-1}$$

Recall:

$$\mathrm{K} = P_{xy}\left(P_{yy} + R\right)^{-1}$$

Ensemble linear filter in practice

- Perturbed observation: add noises to observations to mimic the equation for P_a.
- Ensemble square root filter: use the square roots of P_a ,P_f instead of P_a , P_f.
- Serial ensemble Kalman filter: size of state vector: n, size of observed vector: 1 -> assimilate observations in sequence.

Ensemble linear filter in practice (cont)

- Ensemble transform Kalman filter: size of state vector: 1, size of observed vector: m -> assimilate observations in parallel.

- Local ensemble transform Kalman filter: similar to ETKF but R increases with distances to suppress noises for distant correlations in XY^T.

- Inflation and localization are necessary in large-scale applications.

Extension of the linear filter

To estimate the true state, now we assume a quadratic relation:

$$
\begin{aligned}
\overline{x_a} &= \bar{x} + K_1 \begin{pmatrix} y_1 - \overline{h_1[x]} \\ \vdots \\ y_i - \overline{h_i[x]} \\ \vdots \\ y_m - \overline{h_m[x]} \end{pmatrix}_m \\
&+ K_2 \begin{pmatrix} \langle y_1 - \overline{h_1[x]} \rangle * \langle y_1 - \overline{h_1[x]} \rangle - \overline{\langle y_1 - \overline{h_1[x]} \rangle * \langle y_1 - \overline{h_1[x]} \rangle} \\ \vdots \\ \langle y_i - \overline{h_i[x]} \rangle * \langle y_j - \overline{h_j[x]} \rangle - \overline{\langle y_i - \overline{h_i[x]} \rangle * \langle y_j - \overline{h_j[x]} \rangle} \\ \vdots \\ \langle y_m - \overline{h_m[x]} \rangle * \langle y_m - \overline{h_m[x]} \rangle - \overline{\langle y_m - \overline{h_m[x]} \rangle * \langle y_m - \overline{h_m[x]} \rangle} \end{pmatrix}_{m*m}
\end{aligned}
$$

The additional term can understood as the correction term for the linear filter.

Extension of the linear filter (cont.)

In the simplified form:

$$\overline{x_a} = \bar{x} + \binom{K_1}{K_2}\left(\frac{y - \overline{h[x]}}{\left(y - \overline{h[x]}\right) \otimes \left(y - \overline{h[x]}\right) - \overline{\left(y - \overline{h[x]}\right) \otimes \left(y - \overline{h[x]}\right)}}\right)$$

In general, we can continue with K_3, …, K_n using the triple tensor product, …, the n-order tensor product. Now we understand the Kalman gain K as $(K_1\ K_2\ \ldots\ K_n)^T$

The best estimation of the quadratic filter

$$\begin{pmatrix}K_1\\K_2\end{pmatrix} = \begin{pmatrix}P_{xy}\\P_{xyy}\end{pmatrix}\Bigg[\begin{pmatrix}P_{yy} & P_{yyy}^T\\ P_{yyy} & P_{yyyy}-Vec[P_{yy}]Vec[P_{yy}]^T\end{pmatrix}$$

Here the observation errors are assumed Gaussian. If not, the zero terms should be replaced by the third moments of observation errors.

$$P_a = P_{xx} - \begin{pmatrix}P_{xy}\\P_{xyy}\end{pmatrix}\Bigg[\begin{pmatrix}P_{yy} & P_{yyy}^T\\ P_{yyy} & P_{yyyy}-Vec[P_{yy}]Vec[P_{yy}]^T\end{pmatrix}$$

Others: the fourth moments involving correlations between observation errors and forecast errors.

Linear filter vs. quadratic filter

- Prior moments: the linear filter only uses the first and second prior moments to estimate the analysis. The quadratic filter also calls for the third and fourth moments in estimation.
- Complexity: the size of innovation vector increase from m in the linear filter to m(m+1) in the quadratic filter.

Simplified quadratic filter

We will not use all terms in the tensor product but only the diagonal terms so that we only double the size of innovation vector:

$$\overline{x_a} = \bar{x} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \begin{pmatrix} y - \overline{h[x]} \\ \left(y - \overline{h[x]}\right)^2 - \overline{\left(y - \overline{h[x]}\right)^2} \end{pmatrix}$$

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} P_{xy} \\ P_{xy^2} \end{pmatrix} \left[\begin{pmatrix} P_{yy} & P_{yy^2}^T \\ P_{yy^2} & P_{y^2y^2} - diag[P_{yy}]diag[P_{yy}]^T \end{pmatrix} \right.$$

Simplified ensemble quadratic filter

Ensemble in the model space:

$$X = \frac{1}{\sqrt{K-1}}(x - \bar{x})$$

Ensemble in the observation space:

$$Y = \frac{1}{\sqrt{K-1}}\begin{pmatrix} h[x] - \overline{h[x]} \\ \left|h[x] - \overline{h[x]}\right|^2 - \overline{\left|h[x] - \overline{h[x]}\right|^2} \end{pmatrix}$$

Kalman gain:

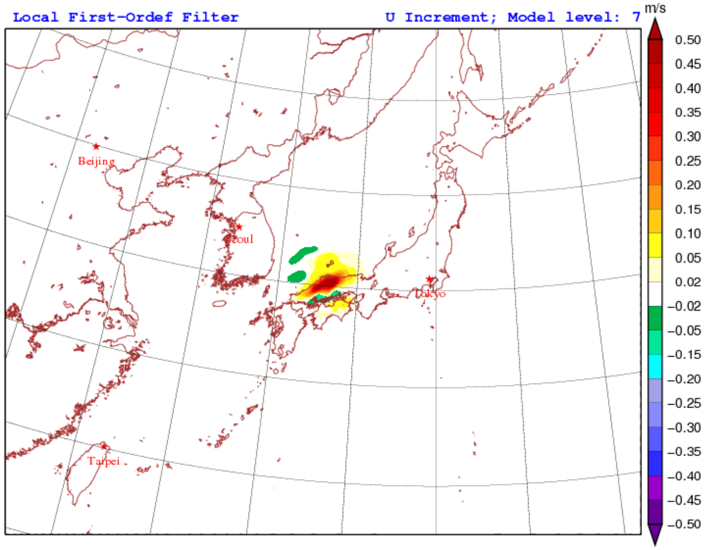
$$K = XY^T\left[YY^T + \begin{pmatrix} R & 0 \\ 0 & 2R\odot R + 4R\odot YY^T \end{pmatrix}\right]^{-1}$$

Simplified ensemble quadratic filter (cont)

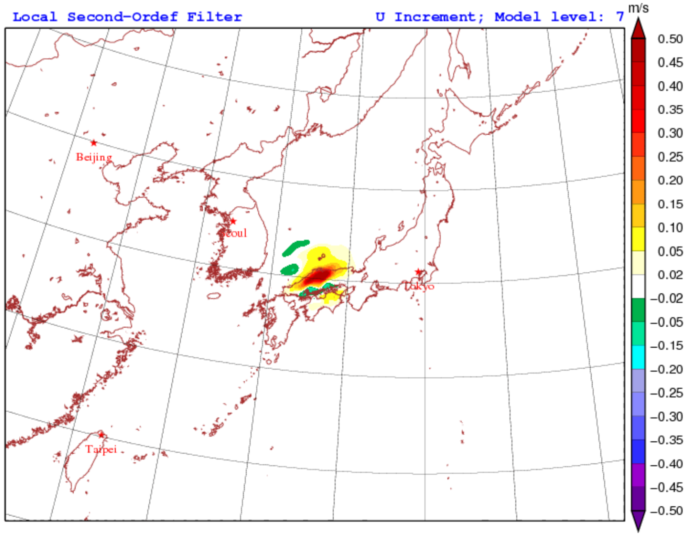
- Similar to the ensemble linear filter, the ensemble quadratic filter can be implemented into many forms.
- Inflation and localization can be applied into large-scale problems as well.

One-obs experiment: u obs of 2 m/s diff

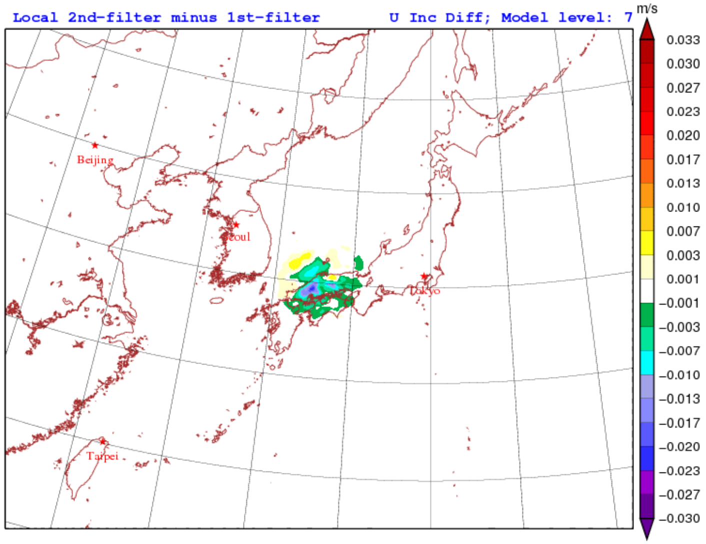
Linear



Quadratic

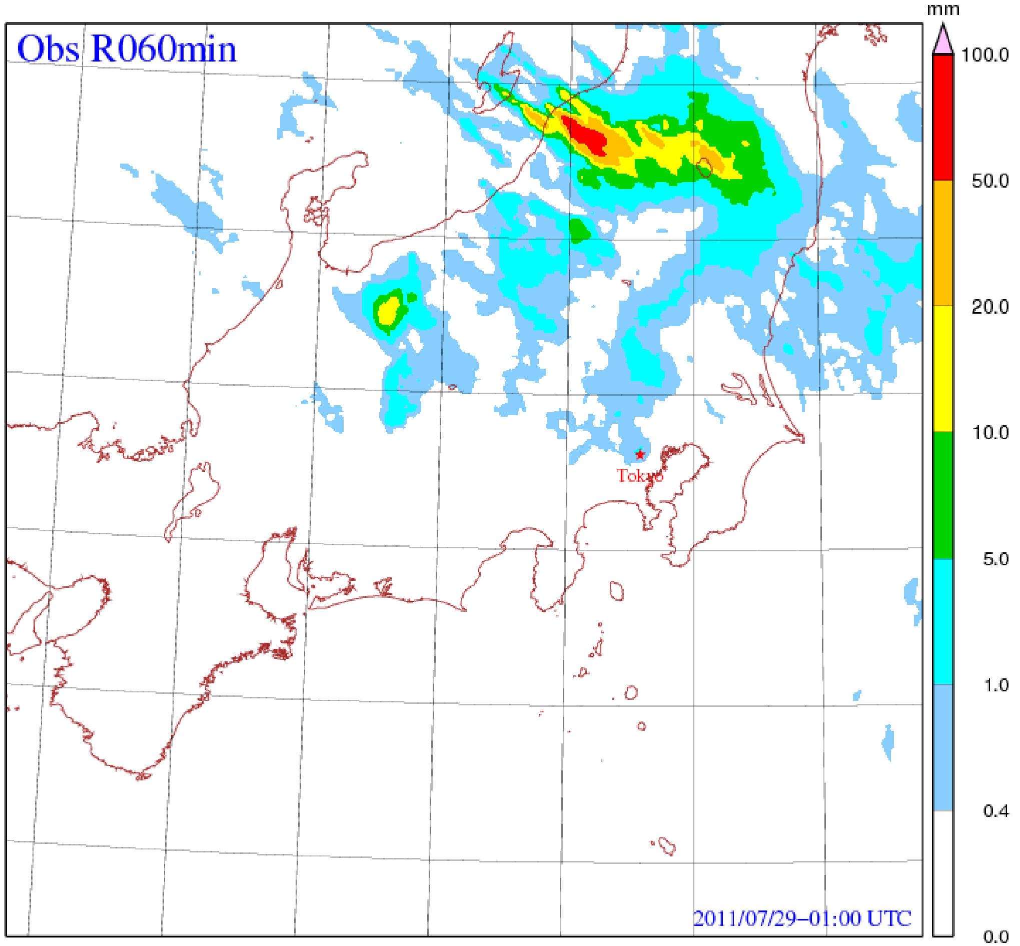


Difference



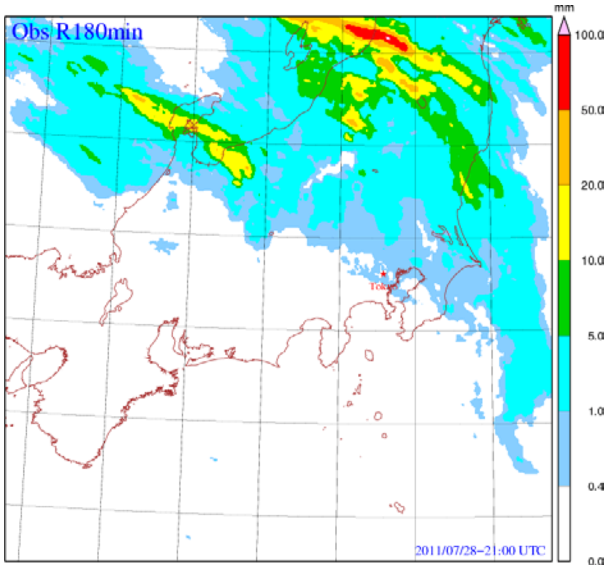
Experimental settings

- Model: NHM
- DA Methods: LETKF forms of 1st and 2nd-order ensemble filters
- Cycle: 3 hours
- Number of cycles: 18
- Domain: 361x289 horizontal grid points, 50 levels
- Resolution: 10 km
- Ensemble: 50 members
- Boundary conditions: GSM forecasts
- Boundary perturbations: JMA's one-week EPS
- Observations: conventional data, GPSs, radar

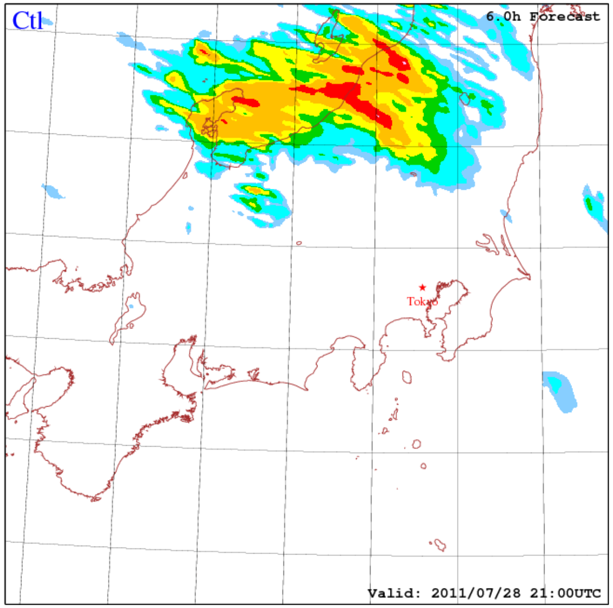


Real obs experiment: 06h forecast

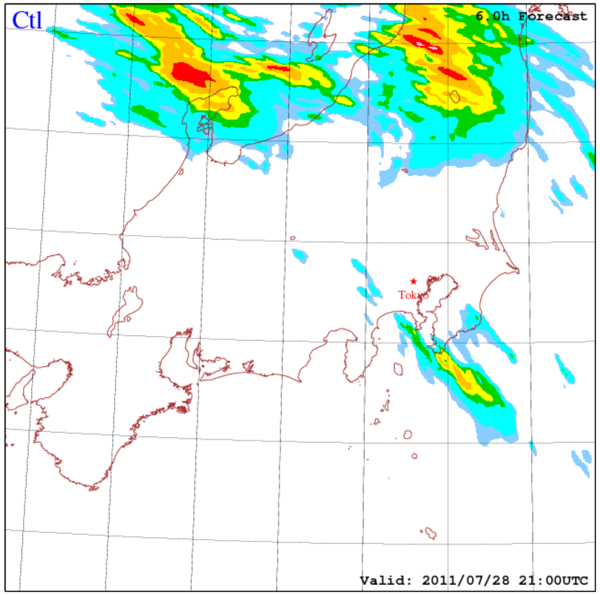
Obs



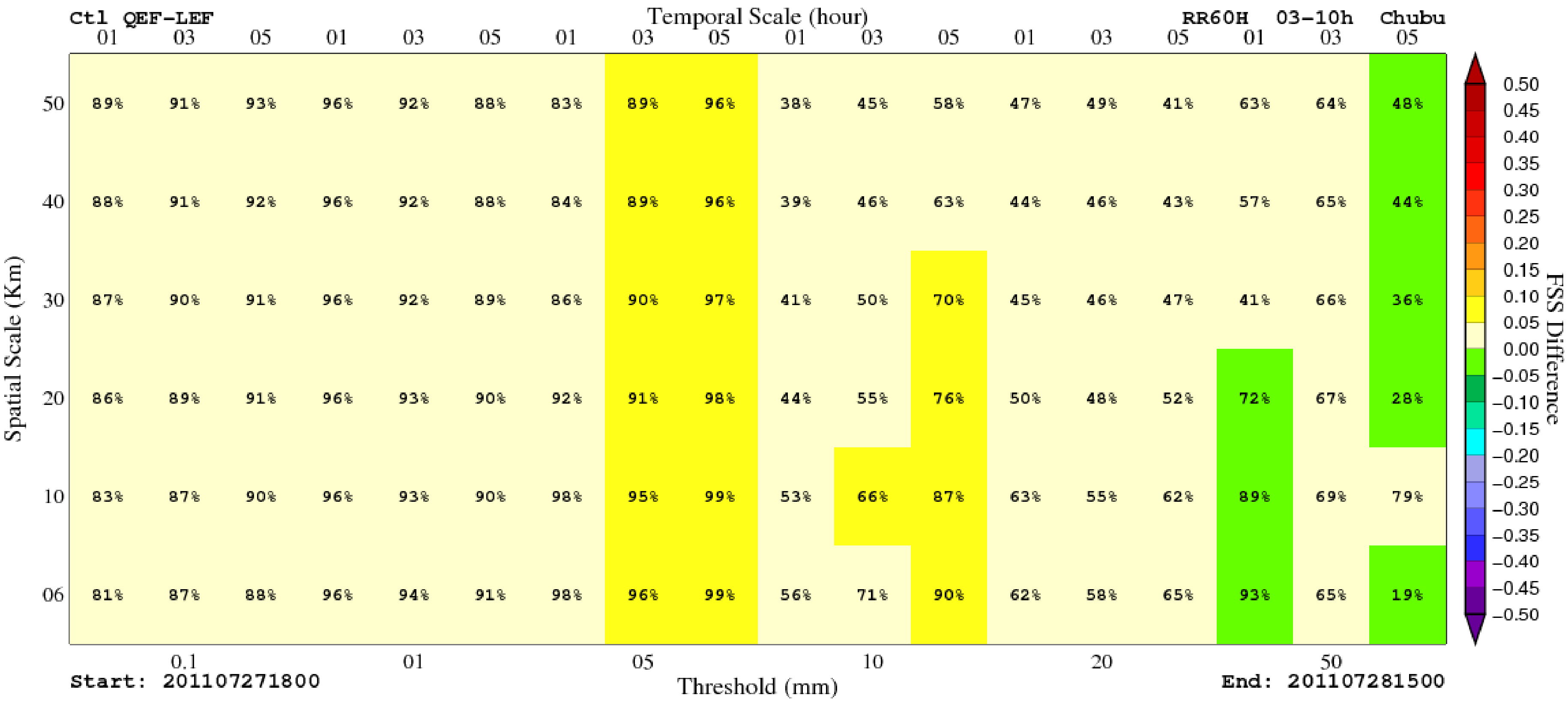
Linear



Quadratic



Statistical verification: Fractional Skill Score



Is it worth running the ensemble quadratic filter?

Summary

- A quadratic filter was extended from the linear filter and was simplified to employ in practice.

- The ensemble realization of this simplified quadratic filter can be easily implemented similar to the existing ensemble methods.

- The test only shows a slightly impact of the quadratic term. More severe tests are needed, especially for non-Gaussian observation errors.